DECISION ANALYSIS

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Summary

This article presents the most fundamental concepts, principles and methods of the scientific discipline called *decision analysis*. After a short introduction to the topic, first, some general concepts of decision analysis are presented. Then well-known decision rules for decision making under uncertainty are described by means of the general concept of a valuation function. Thereby, different degrees of uncertainty are taken into account. In the main section of this article, the most important normative approach to decision making under uncertainty, the so-called Expected Utility Paradigm is presented in detail. Important concepts like the certainty equivalent and the utility function are introduced, the expected utility principle and the general theory as well as the rationality axioms behind are discussed, and, finally, essential empirical results and behavioral extensions of expected utility are pointed out. In another section, the so-called risk-value approach to decision making under uncertainty is presented at length, including both compensatory and lexicographic methods, as well as classic and recent alternative risk-value models. Finally, graphical approaches to decision making under uncertainty like decision trees and influence diagrams are pointed out. All concepts and techniques presented in this article are motivated and illustrated by simple examples.

1. Introduction

Decision analysis is a scientific discipline comprising a collection of principles and methods aiming to help individuals, groups of individuals, or organizations in the performance of difficult decisions. In 1968, Howard announced the existence of this applied discipline to integrate two different streams of research which now are the two pillars upon which most of modern decision analysis rests: normative decision theory and psychological (descriptive) decision theory. The former develops theories of coherent or rational behavior of decision making. Based on an axiomatic footing, certain principles of rationality are developed to which a rational decision maker has to adhere if he or she wants to reach the "best" decision. The latter, psychological decision theory, empirically investigates how (naive) decision-makers really make their decisions and, based on empirical findings, develops descriptive theories about real decision behavior. However, advancements to decision analysis have been made in as different disciplines as mathematics, statistics, probability theory, and artificial intelligence, as well as economics, psychology, and operations research.

Decision problems are characterized by the fact that an individual, a group of individuals, or an organization, the *decision maker* (DM), has the necessity and the opportunity to choose between different *alternatives*. As the name of the discipline suggests, decision analysis decomposes complex decision problems into smaller elements or ingredients of different kinds. Some of these elements are probabilistic in nature, others preferential or value-oriented. Thereby, the presumption is that for decision makers it is easier to make specific statements and judgments on well-identified elements of their decision problems than to make global ad-hoc statements about the quality of the different options between which a choice has to be made. One major task of decision analysis is, at the *structural stage* of the decision making process, to help decision makers to get aware of all the ingredients that have necessarily to be identified in a particular decision problem and to guide them in defining and structuring it. A second important task is, at the *decisional stage* of the decision making process, to develop methods and technologies to reassemble these ingredients so that a choice can be made.

Decision problems play a pervasive role in many economic, political, social, and technological issues but also in personal life. There are many different kinds of such decision problems that can be discerned. Economic decision problems include, e.g., more theoretical problems like the problem of the optimal consumption plan of a household or the optimal production plan of a firm as well as more practical problems like the choice of a house or a car. The manager of a firm has to decide on the optimal location of a new production plant, politicians on the optimal location of a nuclear power plant. An investor has to make a choice on how to invest in different investment options, and engineers on which of different technological alternatives to realize. Given the richness of decision problems, the decision analytic approaches and methods recommended differ from one situation to another. Some general decision analytic concepts, however, can always be identified. Today, decision analysis has evolved into a general thinking framework containing theories, methods, and principles all aiming at a better understanding of any decision-making problem for a better solution.

There are two main problems dealt with in decision analysis: uncertainty and multiple conflicting objectives. *Uncertainty* arises when the quality of the different alternatives of a decision problem depends on states of nature which cannot be influenced by the decision maker and whose occurrence is often probabilistic in nature. These *states of nature* act to produce uncertain possibly different and more or less favorable consequences of each alternative being considered. The sales of a seasonal product like, e.g., ice cream depend on the weather, which cannot be influenced by the producer, and some weather is more favorable for the ice cream sales than another one. *Multiple conflicting objectives* are a typical feature of economic and political decision problems. Any entrepreneur planning a new production plant searches for a location where the wages to be paid are as low as possible and, at the same time, the quality of the personnel is as high as possible. A family house of a certain category should be as cheap as possible and, at the same time, offer a maximum of convenience.

In this article, the general basic concepts, which form the core of modern decision analysis as a scientific discipline, are presented. This presentation includes the basic structure by which, generally, decision problems are characterized and the ingredients, which have to be specified in the structural stage of any practical application. Furthermore, the classical principles, methods, and rules to identify the "best" solution in the decisional stage of a decision problem are developed. Thereby, the emphasis is put on decision making under uncertainty. Decision making with multiple objectives is also touched upon but is treated more detailed in Multiple Criteria Decision Making. All of the concepts will be introduced by providing simple classroom examples.

2. Examples

2.1 Example 1: Decision Problem Under Uncertainty

Assume that Connie is the owner of a bakery and every early Sunday morning she has to prepare some cakes that will hopefully be sold during the day. The cakes contain a special kind of cream that does not stay fresh for more than one day, which means that at the end of the day the unsold cakes must be thrown away. The selling price of a cake is $15.25 and the production cost of a cake is $5.75. Of course, Connie does not know how many cakes will be purchased by customers on a particular Sunday, but by experience, she assumes that the demand will not exceed five cakes. If she wants to have a chance of making any profit at all, she surely should prepare a few cakes. On the other hand if she prepares too many of them it may happen that there will not be enough customers to buy them. The question is how many cakes should she prepare?

This little example is clearly an instance of a decision problem since Connie must decide on the number of cakes to prepare. As a first step, the verbal description of the problem is now represented by a so-called *decision matrix* D defined as follows. Let x denote the number of cakes Connie is going to prepare. Obviously, the value of x is an integer between 0 and 5. So there are six possible values of x, called *alternatives*. Each alternative corresponds to a possible decision by Connie and is associated with a row of the matrix D, i.e. the alternative consisting of making x cakes is associated with the $(x + 1)$-th row of D. Of course, the matrix D has 6 rows.

On the other hand, let y denote the total number of cakes requested by the customers on a particular Sunday. Of course, y is also an integer between 0 and 5 and the value of y is a matter of chance. Each possible value of y is called a *state of nature* and corresponds to a column of the matrix D. More precisely, the state of nature y is associated with the $(y + 1)$-th column of D. So D is a square matrix of dimension 6×6. For $i = 1, ..., 6$ and $j = 1, ..., 6$, let d_{ij} denote the element of the matrix D located at the intersection of the i-th row and the j-th column. Then, by definition, the value of d_{ij} is Connie's profit if she decides to make $(i - 1)$ cakes and the demand of cakes is $(j - 1)$. In this case, it is easy to verify that

$$d_{ij} = \begin{cases} 15.25j - 5.75i - 9.5 & \text{if } i \geq j \\ 9.5(i-1) & \text{if } i < j, \end{cases} \tag{1}$$

which is called the *outcome function*. This leads to the following decision matrix

$$D = \begin{pmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ -5.75 & 9.50 & 9.50 & 9.50 & 9.50 & 9.50 \\ -11.50 & 3.75 & 19.00 & 19.00 & 19.00 & 19.00 \\ -17.25 & -2.00 & 13.25 & 28.50 & 28.50 & 28.50 \\ -23.00 & -7.75 & 7.50 & 22.75 & 38.00 & 38.00 \\ -28.75 & -13.50 & 1.75 & 17.00 & 32.25 & 47.50 \end{pmatrix} \tag{2}$$

In this case, a solution of the decision problem consists of choosing the number of cakes to prepare, which corresponds to the selection of a particular row of the decision matrix. This decision problem obviously is a *decision problem under uncertainty* because the consequences of choosing any number of cakes to prepare depends on the unknown total number of cakes requested by the customers.

2.2 Example 2: Multiple Criteria Decision Problem

Suppose that Brenda and her family want to move to her home city and she is, therefore, looking for a house for her family. Her objectives are:

1. sufficient living space,
2. an acceptable price, and
3. a nice residential area not too far away from downtown.

Furthermore,

4. the house should not be very old, and
5. it should be in good condition.

Assume that Brenda has examined the local daily newspaper and compiled a list of 7 potential houses that seem to meet her objectives. The question is which house she should choose?

As example 1, this example is clearly an instance of a decision problem as Brenda has the necessity and the opportunity to choose between different houses. However, in this case, the problem presents itself differently. Contrary to example 1, first, there is no uncertainty involved (assuming that the prices are more or less fixed and the condition can be verified unequivocally). Second, Brenda is not interested in just one criterion as Connie is in profit. Brenda obviously pursues 5, i.e. multiple objectives where, e.g., price and quality surely are more or less conflicting. Finally, not all of these objectives are already operationalized in a natural way, as it is the case for "profit". "living space", e.g., can be operationalized by the number of rooms of a house as well as by its habitable square meters, and how the "condition" of a house should be operationalized is completely open.

In that example, as a first step of the *structural stage* of the decision making process, each of the more or less *latent variables* corresponding to Brenda's objectives has to be operationalized by a *measurable criterion*. In general, there are many different ways to operationalize a given latent variable. The main goal when operationalizing any latent variable is to minimize the "discrepancy" between that variable and its operationalization. When this discrepancy is "minimized" cannot generally be answered but is a matter of intuition and critical reflection on the competing alternative possibilities to operationalize the given latent variable.

Assume that Brenda has solved the problem of operationalizing all the latent variables corresponding to her five objectives, and, for each of the seven houses, has collected all of their "values". Then, this information can be structured in a two-way table as follows. Let the seven *alternatives* between which Brenda has to choose, denoted by $a_i (i = 1, ..., 7)$, be arranged as the head column of that table, and the five criteria, denoted by $c_i (i = 1, ..., 5)$, as its head line. Then each house is associated with a line of that table where the "values" of the house for the five criteria are summarized. Assume that the information available in Brenda's decision problem is given with Table 1.

Alternative Criterion	**c_1: Number of Rooms**	**c_2: Condition**	**c_3: Age [years]**	**c_4: Price [\$]**	**c_5: Distance to Center [miles]**
a_1: Ash Street	10	good	4	260,000	5
a_2: Beacon Avenue	11	very good	5	240,000	4
a_3: Cambridge Street	7	poor	15	200,000	7
a_4: Davis Square	7	very poor	15	200,000	8
a_5: Exeter Road	7	very poor	20	220,000	8
a_6: Forest Street	9	fair	10	240,000	6
a_7: Glen Road	13	very good	0	320,000	3

Table 1: Brenda's house-buying problem

This table shows that, in general, the different criteria of a multiple criteria decision problem are measured on different scale levels. The first criterion "number of rooms", e.g., is measured on an absolute scale, criteria (3) through (5) on a ratio scale, whereas the second criterion "condition" is only measured on an ordinal scale. Furthermore, the criterion "condition" is not yet quantified by real numbers. An admissible quantification is given by any order-preserving real-valued function, i.e., by any real-valued function assigning real numbers to the "values" of that criterion such that their rank order is respected.

After simply quantifying the condition variable by the first five natural numbers and after rescaling the price variable by the factor 10 000, Brenda's decision problem as given by Table 1 can, as Connie's decision problem, be represented by a *decision matrix* D. As in example 1, each alternative is associated with a row of that matrix. Of course, the matrix D, now, has 7 rows. On the other hand, each criterion c, now, corresponds to a column of the matrix D. Therefore, D is a matrix of dimension 7 × 5. Then, by definition, the value of $d_{ij}(i = 1, ..., 7; j = 1, ..., 5)$ of the matrix is the value of house i on the criterion j. The decision matrix of Brenda's decision problem is given with the matrix

$$D = \begin{pmatrix} 10 & 4 & 4 & 26 & 5 \\ 11 & 5 & 5 & 24 & 4 \\ 7 & 2 & 15 & 20 & 7 \\ 7 & 1 & 15 & 20 & 8 \\ 7 & 1 & 20 & 22 & 8 \\ 9 & 3 & 10 & 24 & 6 \\ 13 & 5 & 0 & 32 & 3 \end{pmatrix} \tag{3}$$

In this case, a solution of the decision problem consists of choosing a house which, as in example 1, corresponds to the selection of a particular row of the decision matrix. This decision problem obviously is a *multiple criteria decision problem* because Brenda simultaneously pursues multiple objectives. It is a *decision problem under certainty* because all the houses are evaluated as if this evaluation were certain.

3. General Concepts

3.5 Decision Matrix

In general, for any kind of decision problem, the decision maker has to choose one alternative out of a set of n mutually exclusive alternatives $a_i(i = 1, ..., n)$. In the case of a *decision problem under uncertainty*, the quality of the different alternatives depends on $m1$ states of nature $z_j(j = 1, ..., m_1)$ which cannot be influenced by the decision maker and lead, for each alternative, to possibly different and more or less favorable consequences. In general, given any state of nature, for each alternative more than one consequence is considered, i.e., after suitable operationalization, the decision maker pursues m_2 criteria $c_k(k = 1, ..., m_2)$. Thereby, a criterion ck is a real-valued function defined on the *set of alternatives*

$$\mathcal{A} = \{a_i, ..., a_i, ..., a_n\} \quad (4)$$

parameterized by the *set of states of nature*

$$\mathcal{Z} = \{z, ..., z_j, ..., z_{m_1}\} \quad (5)$$

i.e. a *criterion* c_k is specified, for every state of nature z_j, by a mapping

$$c_{k,j} : \mathcal{A} \to \mathbb{R}. \quad (6)$$

Multiple criteria decision problems are, in other words, characterized by the fact that for each state of nature z_j the outcome of every alternative a_i is characterized by a m_2-dimensional vector d_{ij} of criteria values with

$$d_{ij} = (c_{1,j}(a_i), ..., c_{k,j}(a_i), ..., c_{m_2,j}(a_i)). \quad (7)$$

This vector denotes the decision maker's *outcome* if he or she chooses alternative a_i and the state of nature z_j happens. The set of all these outcomes can be arranged in the so-called *decision matrix*

$$D = (d_{ij}) \qquad (i = 1, ..., n;\ j = 1, ..., m_1) \quad (8)$$

having n rows and $m1$ columns of vector-valued elements. Each element d_{ij} of this matrix is a m_2- dimensional vector of criteria values. Thereby, it is assumed that any operationalization problem of latent variables already has been solved.

For reasons of simplicity of exposition, in this article, the cases of decision making under uncertainty and with multiple criteria are treated separately. For example, in the case of *decision making under uncertainty*, only one criterion is regarded, and in the case of *multiple criteria decision making*, only one state of nature is considered. This latter case is, therefore, called *multiple criteria decision making under certainty*.

In the case of decision making under uncertainty with only one criterion, i.e. with m_2 =

1, the decision matrix reduces to a $n \times m_1$ matrix where each element d_{ij} is just a single real number. This number indicates the one-dimensional outcome of alternative a_i when state z_j occurs and is called *payoff* in the sequel. In the case of multiple criteria decision making under certainty, i.e. $m_1 = 1$, the decision matrix reduces to a $n \times 1$ matrix where each element d_{i1} is a m_2-dimensional vector of criteria values. This means that also in the case of multiple criteria decision making under certainty the decision matrix reduces to a matrix where each element is just a single real number, i.e. to the $n \times m_2$ matrix where each element is a real number indicating the *value* of an alternative a_i for a certain criterion c_k. This means that, in both cases of decision problems, the starting point for the decisional stage of a decision analysis is a *decision matrix D* of dimension $n \times m$ with real elements, i.e.

$$D = \begin{pmatrix} d_{11} & \cdots & d_{1j} & \cdots & d_{1m} \\ \vdots & & \vdots & & \vdots \\ d_{i1} & \cdots & d_{ij} & \cdots & d_{im} \\ \vdots & & \vdots & & \vdots \\ d_{n1} & \cdots & d_{nj} & \cdots & d_{nm} \end{pmatrix} \quad (9)$$

A first important step at the structural stage of practical decision analyses is to structure the decision problem in the sense of the decision matrix. Thereby, in general, alternatives, objectives, as well as states of nature do not "fall from heaven" but have to be constructed or generated. This process can be very time consuming.

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Biographical Sketches

Hans Wolfgang Brachinger is a Professor in the Department of Quantitative Economics at the University of Fribourg (Switzerland) and holds the Chair of Economic Statistics. After studies in Mathematics, Econometrics, and Statistics at the Universities of Munich, Regensburg, and Tübingen (Germany) he received a master’s degree in Mathematics at the University of Regensburg and doctoral degrees in Statistics and Econometrics of the University of Tübingen. His research areas are decision theory and economic statistics. Dr. Brachinger has published articles in scientific journals such as *The American Economic Review*, *Operations Research Spectrum*, *The Swiss Journal of Economics and Statistics*, as well as the *Journal of Economics and Statistics* and is contributor to the *Encyclopedia of Statistical Sciences* and the *Handbook of Utility*. He has been a referee for many scientific publications. He is a member of the *International Statistical Institute (ISI)*, the *Institute for Operations Research and the Management Science (INFORMS)*, the *Decision Analysis Society (DAS)*, the *Deutsche Statistische Gesellschaft (DStatG)*, the *Deutsche Gesellschaft für Operations Research* (GOR) as well as of the *Gesellschaft für Wirtschafts- und Sozialwissenschaften - Verein für Socialpolitik* (VS) where he is fellow of the *Ausschuss für Ökonometrie* as well as of the *Sozialwissenschaftlicher Ausschuss.* He has been co-organizer and chairman of many national and international scientific conferences.

Paul-André Monney is Associate Professor of Statistics in the Department of Quantitative Economics at the University of Fribourg, Switzerland. He received a Doctoral Degree in Mathematics and the Venia Legendi in Statistics from the University of Fribourg. He has done extensive research in Theoretical Computer Science and Statistics, in particular the Dempster-Shafer Theory of Evidence. Dr. Monney has numerous publications, including articles in scientific journals such as *Artificial Intelligence*, *International Journal of Approximate Reasoning*, *Journal of Computational and Applied Mathematics* and *Zeitschrift für Operations Research*. He is co-author of *A Mathematical Theory of Hints,* and an author of *An Approach to the Dempster-Shafer Theory of Evidence,* a book presenting a new perspective on the Dempster-Shafer Theory of Evidence. Dr. Monney has served as a member of the program committee, or as a referee, for several international conferences.